A note on shock flow in a channel

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SUMMARY

A shock wave is generated by a uniform compressive piston motion and passes into a channel of slowly varying cross-section. A relation in closed form is obtained between shock strength and the area of the channel and is used to discuss converging cylindrical and spherical shocks.

1. INTRODUCTION

Using a linearized theory based on small area variations, Chester (1953, 1954) discussed the motion of a uniform shock wave passing through a two-dimensional channel composed of two uniform cross-sections separated by a section of slowly varying cross-section. The fluid in front of the shock was at rest; initially, the flow behind was isentropic, but when the shock entered the transition section the shock strength was altered and the subsequent flow was not isentropic. The disturbance depended on the area but not on the shape of the cross-section; it consisted of two perturbations: a permanent disturbance due directly to the area change, and a transient disturbance, propagated with sonic velocity relative to the flow behind the shock, due to the reflection of the permanent disturbance from the shock. The pressure change behind the shock was determined. Chisnell (1957) integrated this first-order relation to obtain a functional relation between the channel area and the shock strength, and used particular channel shapes to discuss converging cylindrical and spherical shocks. His results agree closely with previous similarity solutions obtained by Guderly (1942) and Butler (1945).

In Chester's solution, the shock has, so to speak, come from infinity, and there is no possibility for reflections to occur upstream of the shock. In a discussion of the non-steady flow in a channel of slowly varying cross-section, linearized with respect to small area variations, the present author obtained Chester's solution on the basis of one-dimensional theory. A simple but interesting generalization is to assume that a shock is produced by a uniform compressive piston motion; when it passes into a channel of varying cross-section, there are three distinct contributions to the disturbance: a permanent disturbance due to the area change, a transient disturbance due to reflection from the shock, and also a transient disturbance due to reflection from the piston. This problem was solved by the author (Gundersen 1958) for the case of a slowly converging or diverging cross-section, and an expression was given for the pressure change directly behind the shock. This differential relation is integrated here in closed form to give a functional relation between channel area and shock strength. Since the shock strength is uniform over its area for the symmetrical converging cylindrical and spherical flows, this result can be utilized to discuss such flows.

2. The relation between channel area and shock strength

The fluid is assumed perfect with constant specific heats. Denote by γ the adiabatic index, by P_0 and c_0 the pressure and sound velocity in the gas at rest in front of the shock, by w the shock velocity, and by u_2, P_2 and c_2 the fluid velocity, pressure and sound velocity behind the shock. Let w_2 be the shock velocity relative to the flow behind it, so that $w_2 = w - u_2$, and let M_0, M_1 and m denote the following Mach numbers, $M_0 = w/c_0$, $M_1 = w_2/c_2$, $m = u_2/c_2$. Then from the usual formulae for a normal shock:

$$w_2 = (\gamma-1+2M_0^{-2})\frac{w}{\gamma+1}, \qquad M_1^2 = \frac{\gamma-1+2M_0^{-2}}{2\gamma-(\gamma-1)M_0^{-2}},$$

$$u_2 = \frac{2w(1-M_0^{-2})}{\gamma+1}, \qquad m = \frac{2(1-M^2)}{(\gamma+1)M_1}, \qquad P_2 = (2\gamma M_0^2-\gamma+1)\frac{P_0}{\gamma+1}.$$

It is convenient to express the cross-section of the tube in the form $E(x) = E_0+E_1(x)$, where E_0 is the original uniform cross-sectional area. In terms of M_0, M_1 and m, the pressure perturbation directly behind the shock, for $E_1(x) = kx$ (where k is a constant, positive for a diverging section and negative for a converging section) and when the shock is produced by a uniform compressive piston motion, is

$$\bar{P}_2 = -(P_2-P_0)E_1E_0^{-1}K. \tag{1}$$

There is a small error in the previously presented K, which is due to the omission of a factor of 2 in the next to last equation on page 564 of the previous paper, which should have read

$$\frac{\bar{\rho}_0}{\rho_0} = 2\left[\frac{u_0-(1-\theta)W_0}{W_0(u_0-W_0)}\right]\epsilon(t),$$

so that equation (8.2) should have read

$$\bar{s}_0 = 2c_v(\gamma-1)\left\{\frac{K_2}{c_0}-\left[\frac{u_0-(1-\theta)W_0}{(u_0-W_0)W_0}\right]\right\}\epsilon(t).$$

This last expression also appears on page 575 with different subscripts, and the correct expression in terms of the parameters M_1, M_1 and m is

$$\bar{s}_2 = c_v(\gamma-1)w^{-1}\{4(1-M_1^2)[(\gamma+1)(1-M_0^{-2})]^{-1}+ \\ +m^2-2+2(\gamma-1)(\gamma-1+2M_0^{-2})^{-1}\}.$$

When this expression is substituted into the defining equation for K_6 on page 576, the correct K of (1) is defined by

$$2(\gamma-1+2M_0^{-2})(\gamma+1)^{-1}K^{-1} \\ = 2M_1^2+mM_1(1-M_0^{-2})+(1+M_0^{-2})-(\gamma-1+2M_0^{-2})(2\gamma)^{-1}\times \\ \times\{4(1-M_1^2)[(\gamma+1)(1-M_0^{-2})]^{-1}+m^2-2+ \\ +2(\gamma-1)(\gamma-1+2M_0^{-2})^{-1}\},$$

and is a monotonic decreasing function of the shock strength. For $\gamma = 1{\cdot}4$, K varies from 0·500 for weak shocks to an asymptotic limit of 0·259 for strong shocks. In Chester's solution, the variation of the corresponding K is from 0·5 to 0·394.

To facilitate comparison, Chisnell's notation will now be utilized when possible. If the initial shock strength is defined by $P_2/P_1 = z$, (1) takes the form:

$$-\frac{1}{E}\frac{dE}{dz} = \frac{1}{(z-1)K(z)}$$

$$= \frac{(\gamma^2-1)(z-1)}{2\gamma^2 z[(\gamma-1)z+\gamma+1]} + \frac{(\gamma-1)[(\gamma+1)z+\gamma-1]}{2\gamma^2 z(z-1)} + $$

$$+ \frac{(\gamma+1)(z+3)}{2(z-1)[(\gamma-1)z+\gamma+1]}. \quad (2)$$

Integration gives the relation between area and shock strength, but as Chisnell points out, for the case of two uniform channels connected by a transition area of varying cross-section, the result gives the average strength of the shock after the area change only if the total change is small. For large area changes, it will be only an approximation to the average strength.

The integration of (2) gives

$$Ef(z) = \text{const.},$$

where

$$f(z) = (z-1)^2 z^{(1-\gamma)/2\gamma}[(\gamma-1)z+\gamma+1]^{(3-\gamma)/2(\gamma-1)}.$$

For weak shocks (z nearly unity), $f(z)$ behaves like $(z-1)^2$ for $\gamma = 1{\cdot}4$, i.e. the acoustic result that the strength of the disturbance varies inversely as the square root of the area of the disturbance.

As pointed out by Chisnell, the above results are applicable to converging cylindrical and spherical shocks if channel areas are proportional to R or R^2, respectively, and R is the distance of the shock from its axis of symmetry.

Near the point of collapse of the symmetrical shocks, z is very large, and $f(z)$ behaves like $z^{1/K}$, where K, the asymptotic limit of $K(z)$, is

$$K = \frac{2\gamma(\gamma-1)}{(2\gamma-1)(\gamma+1)}.$$

Hence, for cylindrical and spherical shocks near their axes of symmetry, the strength is proportional to R^{-K} and R^{-2K}, respectively. The following table compares the values of K with those obtained by Chisnell for the corresponding number in his paper.

	Chisnell	This paper
$\gamma = \frac{6}{5}$	0·326223	0·155844
$\gamma = \frac{7}{5}$	0·394141	0·259259
$\gamma = \frac{5}{3}$	0·450850	0·357143

For $\gamma = \frac{6}{5}, \frac{7}{5}, \frac{5}{3}$, the numbers of the present paper are about 52%, 34% and 21%, respectively, smaller than the numbers given by Chisnell, i.e. the perturbations reflected from the piston reduce the ultimate shock strength, near the point of collapse, to a value less than in the Chisnell determination.

References

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